LESSON 4.5b

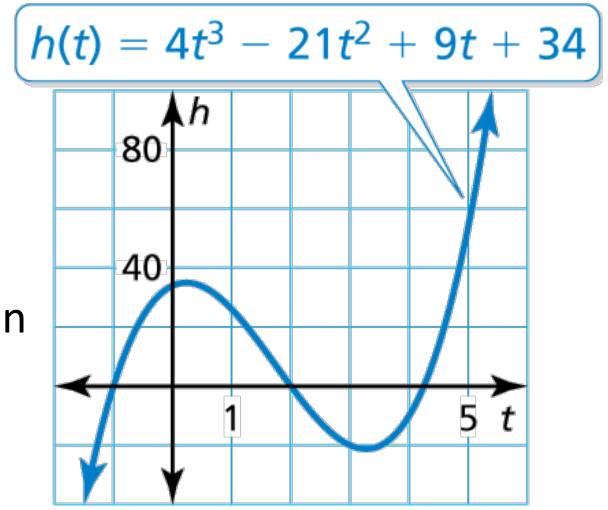
Rational Roots Theorem

Today you will:

- Use the Rational Roots Theorem to find potential roots of a polynomial.
- Practice using English to describe math processes and equations

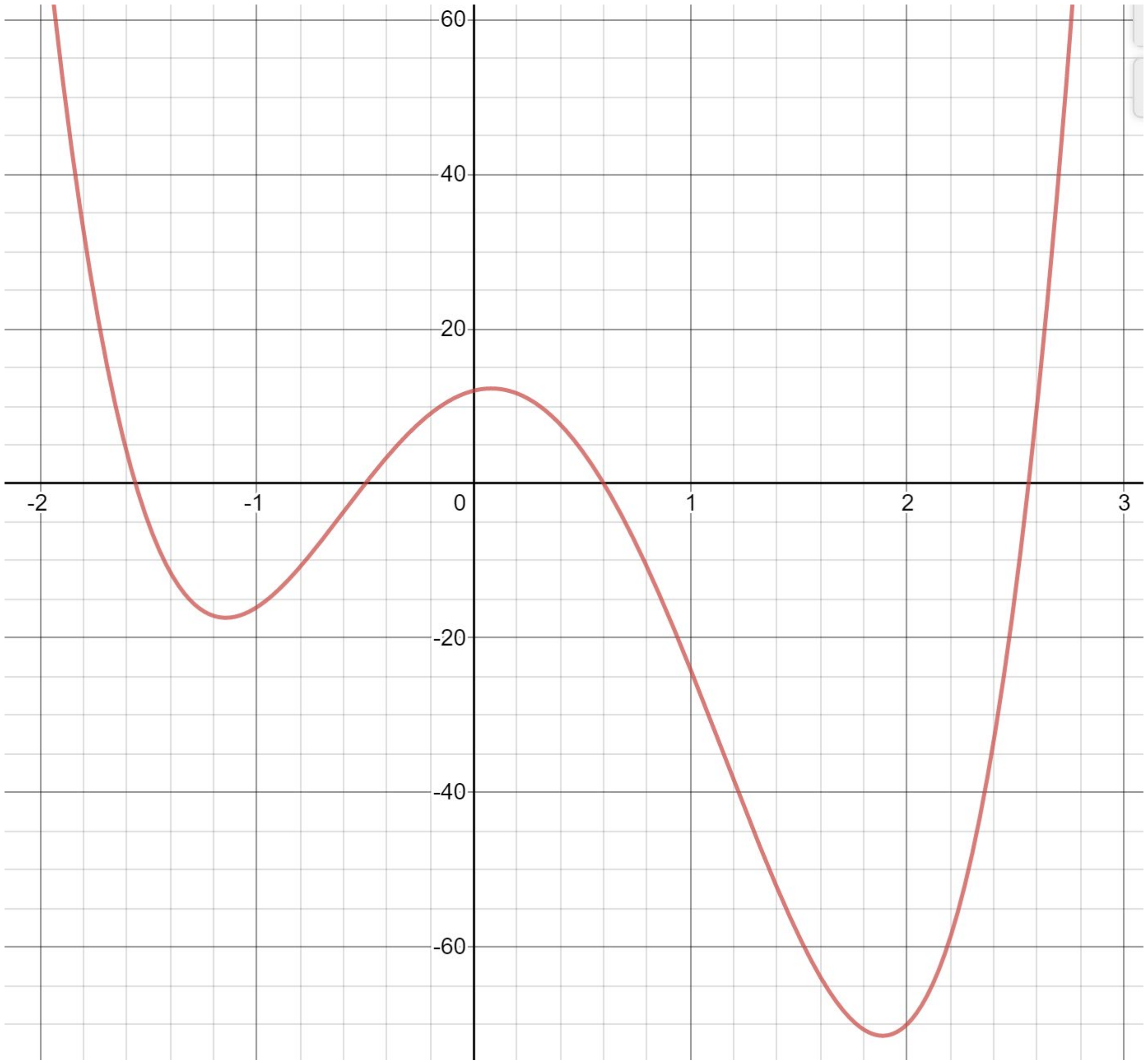
The Remainder and Factor Theorems are our friends:

- If we have a good guess at one root of the function, we can get going and factor it.
- We can get that guess by graphing. For example:
- Here we see apparent roots at -1 and 2, with the 3rd between 4 & 5.
- We can determine if -1 is a root by using synthetic division and divide the function by -1 … if the remainder is zero we have found a root.



What if we cannot see obvious candidates?

- What do we do then?
- How do we get started?



Core Vocabulary:

- Rational Roots Theorem, p. 191

If the polynomial has integer coefficients, you can find the ***possible*** roots of a polynomial using the leading coefficient and the constant (ending) term.

Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$.

SOLUTION

The polynomial $f(x) = x^3 - 8x^2 + 11x + 20$ is not easily factorable. Begin by using the Rational Root Theorem.

ANOTHER WAY

You can use direct substitution to test possible solutions, but synthetic division helps you identify other factors of the polynomial.

Step 1 List the possible rational solutions. The leading coefficient of $f(x)$ is 1 and the constant term is 20. So, the possible rational solutions of $f(x) = 0$ are

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{20}{1}.$$

Step 2 Test possible solutions using synthetic division until a solution is found.

Test $x = 1$:

$$\begin{array}{r|rrrr} 1 & 1 & -8 & 11 & 20 \\ & & 1 & -7 & 4 \\ \hline & 1 & -7 & 4 & 24 \end{array}$$

Test $x = -1$:

Step 3 Factor completely using the result of the synthetic division.

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$(x + 1)(x^2 - 9x + 20) = 0$ Write as a product of factors.

$(x + 1)(x - 4)(x - 5) = 0$ Factor the trinomial.

So, the solutions are $x = -1$, $x = 4$, and $x = 5$.

Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$.

SOLUTION

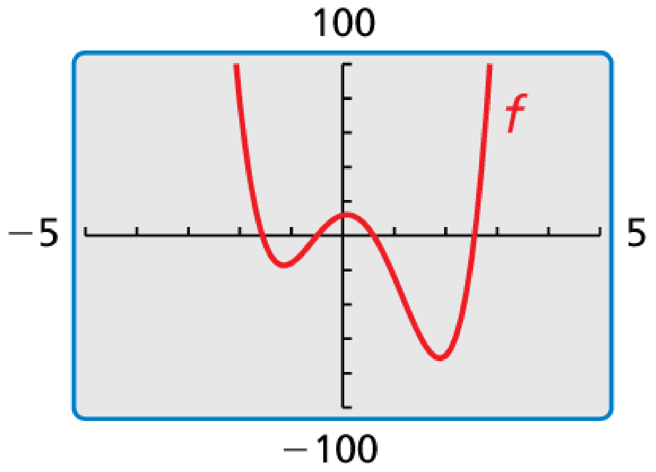
Step 1 List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1},$

$$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$$

Step 2 Choose reasonable values from the list above to test using the graph of the function. For f, the values

$$x = -\frac{3}{2},\ x = -\frac{1}{2},\ x = \frac{3}{5}, \text{ and } x = \frac{12}{5}$$

are reasonable based on the graph shown at the right.



Step 3 Test the values using synthetic division until a zero is found.

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 10 & -11 & -42 & 7 & 12 \\ & & -15 & 39 & \frac{9}{2} & -\frac{69}{4} \\ \hline & 10 & -26 & -3 & \frac{23}{2} & -\frac{21}{4} \end{array}$$

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 10 & -11 & -42 & 7 & 12 \\ & & -5 & 8 & 17 & -12 \\ \hline & 10 & -16 & -34 & 24 & 0 \end{array}$$

$-\frac{1}{2}$ is a zero.

Step 4 Factor out a binomial using the result of the synthetic division.

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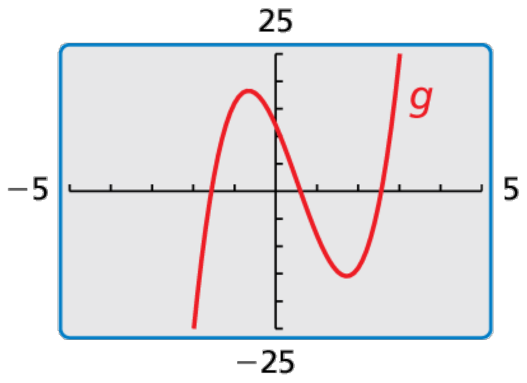
$$f(x) = \left(x + \frac{1}{2}\right)(10x^3 - 16x^2 - 34x + 24)$$ Write as a product of factors.

$$= \left(x + \frac{1}{2}\right)(2)(5x^3 - 8x^2 - 17x + 12)$$ Factor 2 out of the second factor.

$$= (2x + 1)(5x^3 - 8x^2 - 17x + 12)$$ Multiply the first factor by 2.

Step 5 Repeat the steps above for $g(x) = 5x^3 - 8x^2 - 17x + 12$. Any zero of g will also be a zero of f. The possible rational zeros of g are:



$$x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$$

The graph of g shows that $\frac{3}{5}$ may be a zero. Synthetic division shows that $\frac{3}{5}$ *is* a zero and $g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4)$.

It follows that:

$f(x) = (2x + 1) \bullet g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)$

Step 6 Find the remaining zeros of f by solving $x^2 - x - 4 = 0$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)}$$ Substitute 1 for a, −1 for b, and −4 for c in the Quadratic Formula.

$$x = \frac{1 \pm \sqrt{17}}{2}$$ Simplify.

The real zeros of f are $-\frac{1}{2}$, $\frac{3}{5}$, $\frac{1 + \sqrt{17}}{2} \approx 2.56$, and $\frac{1 - \sqrt{17}}{2} \approx -1.56$.

Homework

Pg 194, #21-40